The deformations of nondegenerate constant Poisson bracket with even and odd deformation parameters

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Abstract

We consider Poisson superalgebras with constant nondegenerate bracket realized on the smooth Grassmann-valued functions with compact supports in $\mathbb{R}^n$. The deformations with even and odd deformation parameter of these superalgebras are presented for $n \geq 4$.

1 Introduction.

The hope to construct the quantum mechanics on nontrivial manifolds is connected with geometrical or deformation quantization [1] - [4]. The functions on the phase space are associated with the operators, and the product and the commutator of the operators are described by associative *-product and *-commutator of the functions. These *-product and *-commutator are the deformations of usual product and of usual Poisson bracket. The deformations of Poisson (anti)bracket was considered in many publications for different spaces of functions. In [5], the problem is considered for the Poisson superalgebra on the superspace of polynomials and for the antibracket. Purely Grassmannian case is considered in [6]. Bosonic Poisson algebra realized on smooth functions was considered in [7].

The result depends on chosen function space (see [5] for more details).

It occurred, that there exist odd second cohomologies with coefficients in adjoint representation. It is natural to look for the deformations associated with these odd cohomologies and having the odd deformation parameter [5], [8].

1. Deformations of topological Lie superalgebras. We recall some concepts concerning formal deformations of algebras (see, e.g., [13]), adapting them to the case of topological Lie superalgebras. Let L be a topological Lie superalgebra over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with Lie superbracket $\{\cdot,\cdot\}$, $\mathbb{K}[[\hbar^2]]$ be the ring of formal power series in $\hbar^2$ over $\mathbb{K}$, and $L[[\hbar^2]]$ be the $\mathbb{K}[[\hbar^2]]$-module of formal power series in $\hbar^2$ with coefficients in L. We endow both $\mathbb{K}[[\hbar^2]]$ and $L[[\hbar^2]]$ by the direct-product topology. The grading of L naturally determines a grading

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of $L[[\hbar^2]]$: an element $f = f_0 + \hbar^2 f_1 + \ldots$ has a definite parity $\varepsilon(f)$ if $\varepsilon(f) = \varepsilon(f_j)$ for all $j = 0, 1, ...$ Every p-linear separately continuous mapping from L^p to L (in particular, the bracket $\{\cdot,\cdot\}$) is uniquely extended by $\mathbb{K}[[\hbar^2]]$-linearity to a p-linear separately continuous mapping over $\mathbb{K}[[\hbar^2]]$ from $L[[\hbar^2]]^p$ to $L[[\hbar^2]]$. A (continuous) formal deformation of L is by definition a $\mathbb{K}[[\hbar^2]]$-bilinear separately continuous Lie superbracket $C(\cdot,\cdot)$ on $L[[\hbar^2]]$ such that $C(f,g) = \{f,g\} \mod \hbar^2$ for any $f, g \in L[[\hbar^2]]$. Obviously, every formal deformation C is expressible in the form

$$C(f,g) = \{f,g\} + \hbar^2 C_1(f,g) + \hbar^4 C_2(f,g) + \ldots, \quad f,g \in L, \tag{1.1}$$

where C_j are separately continuous skew-symmetric bilinear mappings from $L \times L$ to L (2-cochains with coefficients in the adjoint representation of L). Formal deformations C^1 and C^2 are called equivalent if there is a continuous $\mathbb{K}[[\hbar^2]]$-linear operator $T : L[[\hbar^2]] \to L[[\hbar^2]]$ such that $TC^1(f,g) = C^2(Tf,Tg)$, $f, g \in L[[\hbar^2]]$ and $T = \mathrm{id} + \hbar^2 T_1$. The problem of finding formal deformations of L is closely related to the problem of computing Chevalle–Eilenberg cohomology of L with coefficients in the adjoint representation of L. Let $\mathcal{C}_p(L)$ denote the space of p-linear skew-symmetric separately continuous mappings from L^p to L (the space of p-cochains with coefficients in the adjoint representation of L). The space $\mathcal{C}_p(L)$ possesses a natural $\mathbb{Z}_2$-grading: by definition, $M_p \in \mathcal{C}_p(L)$ has the definite parity $\epsilon(M_p)$ if the relation

$$\varepsilon(M_p(f_1,\ldots,f_p)) = \varepsilon(M_p) + \varepsilon(f_1) + \ldots + \varepsilon(f_1)$$

holds for any $f_j \in L$ with definite parities $\epsilon(f_j)$. Since a Lie superbracket is always even, all C_j in the expansion (1.1) should be even 2-cochains. The differential d_p^{ad} is defined to be the linear operator from $\mathcal{C}_p(L)$ to $\mathcal{C}_{p+1}(L)$ such that

$$d_p^{\mathrm{ad}} M_p(f_1,...,f_{p+1}) = -\sum_{j=1}^{p+1} (-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{1,j-1}+\varepsilon(f_j)\varepsilon_{M_p}} \{f_j, M_p(f_1,...,\hat{f}_j,...,f_{p+1})\} -$$
$$- \sum_{i<j} (-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{i+1,j-1}} M_p(f_1,...f_{i-1},\{f_i,f_j\},f_{i+1},...,\hat{f}_j,...,f_{p+1}),$$

for any $M_p \in \mathcal{C}_p(L)$ and $f_1, \ldots f_{p+1} \in L$ having definite parities. Here the hat means that the argument is omitted and the notation $|\varepsilon(f)|_{i,j} = \sum_{l=i}^{j} \varepsilon(f_l)$ has been used. Writing the Jacobi identity for a deformation C of the form (1.1),

$$(-1)^{\varepsilon(f)\varepsilon(h)} C(f, C(g,h)) + \mathrm{cycle}(f,g,h) = 0,$$

and taking the terms of the order $\hbar^2$, we find that

$$d_2^{\mathrm{ad}} C_1 = 0.$$

Thus, the first order deformations of L are described by 2-cocycles of the differential d^{ad}.

2. Poisson superalgebra. Let $\mathcal{D}(\mathbb{R}^k)$ denote the space of smooth $\mathbb{K}$-valued functions with compact support on $\mathbb{R}^k$. This space is endowed by its standard topology: by definition, a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^k)$ converges to $\varphi \in \mathcal{D}(\mathbb{R}^k)$ if $\partial^\lambda \varphi_k$ converge uniformly to $\partial^\lambda \varphi$ for every multi-index λ, and the supports of all φ_k are contained in a fixed compact set. We set

$$D_{n_+}^{n_-} = \mathcal{D}(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad E_{n_+}^{n_-} = C^\infty(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-},$$

where $\mathbb{G}^{n_-}$ is the Grassmann algebra with n_- generators. The generators of the Grassmann algebra (resp., the coordinates of the space $\mathbb{R}^{n_+}$) are denoted by ξ_α, $\alpha = 1,\dots,n_-$ (resp., x_i, $i = 1,\dots,n_+$). We shall also use collective variables z_A which are equal to x_A for $A = 1,\dots,n_+$ and are equal to ξ_{A-n_+} for $A = n_+ + 1,\dots,n_+ + n_-$. The spaces $D^{n_-}_{n_+}$ and $E^{n_-}_{n_+}$ possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element f of these spaces is denoted by $\varepsilon(f)$. We also set $\varepsilon_A = 0$ for $A = 1,\dots,n_+$ and $\varepsilon_A = 1$ for $A = n_+ + 1,\dots,n_+ + n_-$.

Let $\partial/\partial z_A$ and $\overleftarrow{\partial}/\partial z_A$ be the operators of the left and right differentiation. The Poisson bracket is defined by the relation

$$\{f,g\}(z) = \sum_{A,B} f(z)\frac{\overleftarrow{\partial}}{\partial z_A}\omega^{AB}\frac{\partial}{\partial z_B}g(z) = -(-1)^{\varepsilon(f)\varepsilon(g)}\{g,f\}(z), \tag{1.2}$$

where the symplectic metric $\omega^{AB} = (-1)^{\varepsilon_A\varepsilon_B}\omega^{BA}$ is a constant invertible matrix. For definiteness, we choose it in the form

$$\omega^{AB} = \begin{pmatrix} \omega^{ij} & 0 \\ 0 & \lambda_\alpha\delta^{\alpha\beta} \end{pmatrix}, \quad \lambda_\alpha = \pm 1,\ i,j = 1,...,n_+,\ \alpha,\beta = 1,...,n_-$$

where ω^{ij} is the canonical symplectic form (if $\mathbb{K} = \mathbb{C}$, then one can choose $\lambda_\alpha = 1$). The nondegeneracy of the matrix ω^{AB} implies, in particular, that n_+ is even. The Poisson superbracket satisfies the Jacobi identity

$$(-1)^{\varepsilon(f)\varepsilon(h)}\{f,\{g,h\}\}(z) + \mathrm{cycle}(f,g,h) = 0, \quad f,g,h \in E^{n_-}_{n_+}. \tag{1.3}$$

By Poisson superalgebra $\mathcal{P}^{n_-}_{n_+}$, we mean the space $D^{n_-}_{n_+}$ with the Poisson bracket (1.2) on it. The relations (1.2) and (1.3) show that this bracket indeed determines a Lie superalgebra structure on $D^{n_-}_{n_+}$.

The integral on $D^{n_-}_{n_+}$ is defined by the relation

$$\bar{f} \stackrel{\text{def}}{=} \int dz\, f(z) = \int_{\mathbb{R}^{n_+}} dx \int d\xi\, f(z),$$

where the integral on the Grassmann algebra is normed by the condition $\int d\xi\,\xi_1\dots\xi_{n_-} = 1$.

Introduce the superalgebra $Z^{n_-}_{n_+}$, $D^{n_-}_{n_+} \subset Z^{n_-}_{n_+} \subset E^{n_-}_{n_+}$, $Z^{n_-}_{n_+} = D^{n_-}_{n_+} \oplus \mathcal{C}_{D^{n_-}_{n_+}}(E^{n_-}_{n_+})$, where $\mathcal{C}_{D^{n_-}_{n_+}}(E^{n_-}_{n_+})$ is centralizer of $D^{n_-}_{n_+}$ in $E^{n_-}_{n_+}$.

Introduce the notation

$$S^m_n \stackrel{def}{=} E^m_n / Z^m_n. \tag{1.4}$$

3. Poisson superalgebra with outer odd parameters. Below we consider the case, where the functions and multilinear forms may depend on outer odd parameters θ_i, where θ-s belong to some supercommutative associative superalgebra. For simplicity we consider the case $\theta_i \in \mathbb{G}^k$. Thus we consider a colored algebra $\mathbf{D} = \mathbb{G}^k \otimes D^{n_-}_{n_+}$, with $(\mathbb{Z}_2)^2$ grading, namely the grading of element $\theta \otimes f$ is $(\varepsilon_1(\theta), \varepsilon_2(f))$.

Below we consider $\mathbf{D}$ as a Lie superalgebra with the parity $\varepsilon = \varepsilon_1 + \varepsilon_2$. One can easily check that such consideration is selfconsistent (see also [9] and discussion on Necludova and Sheunert theorems in [10].

Analogously we introduce superspaces and superalgebras

$$\begin{aligned}\mathbf{E} &= \mathbb{G}^k \otimes E_{n_+}^{n_-} \\ \mathbf{Z} &= \mathbf{D} \oplus \mathcal{C}_{\mathbf{D}}(\mathbf{E}) \\ \mathbf{S} &= \mathbb{G}^k \otimes S_{n_+}^{n_-}\end{aligned} \tag{1.5}$$

In the following sections we will consider two cases separately:

1. $k = 1$, there exists only one odd parameter θ, $\theta^2 = 0$.
2. $k > 1$.

All these parameters θ are supposed to be generating elements of $\mathbb{G}^k$, i.e. they are odd and satisfy the following condition:

if $\theta y = 0$ for some $y \in \mathbb{G}^k$ then $y = \theta z$ for some $z \in \mathbb{G}^k$.

4. Jacobiators. Let p, q be antisymmetric bilinear forms. Here and below Jacobiators are defined as follows:

$$\begin{aligned}J(p,q) &\stackrel{def}{=} (-1)^{\varepsilon(f)\varepsilon(h)}\left(p(q(f,g),h) + q(p(f,g),h)\right) + \text{cycle}(f,g,h), \\ J(p,p) &\stackrel{def}{=} (-1)^{\varepsilon(f)\varepsilon(h)} p(p(f,g),h) + \text{cycle}(f,g,h).\end{aligned}$$

Evidently, $J(p,q) \in \mathcal{C}_3(\mathbf{D},\mathbf{D})$, if $p, q \in \mathcal{C}_2(\mathbf{D},\mathbf{D})$.

If $m_0(f,g) = \{f,g\}$ then $-(-1)^{\varepsilon(f)\varepsilon(h)} J(p,m_0) = d_2^{\mathrm{ad}} p$.

5. Sign rules. We use here the following sign rules for factorization the odd parameters

$$M_n(\theta f_1, f_2, ..., f_n) = (-1)^{\varepsilon(\theta)\varepsilon(M_n)}\theta M_n(f_1, ..., f_n) \tag{1.6}$$

It follows from this sign rule and superantisymmetry of the form M_n that‘

$$M_n(f_1, \theta f_2, ..., f_n) = M_n(f_1\theta, f_2, ..., f_n) \tag{1.7}$$

$$M_n(f_1, f_2, ..., f_n\theta) = M_n(f_1, f_2, ..., f_n)\theta. \tag{1.8}$$

2 1 odd deformation parameter

Let θ be the odd deformation parameter, $\varepsilon(\theta) = 1$.

Let deformation $C(f,g)$ has the form

$$C(f,g) = C_0(f,g) + \theta C_1(f,g),$$

where $\varepsilon(C_0) = 0$, $\varepsilon(C_1) = 1$.

Jacoby identity gives

$$0 = J(C,C) = J(C_0,C_0) + J(C_0,\theta C_1), \tag{2.1}$$

which implies

$$\begin{aligned}J(C_0,C_0) &= 0 \\ J(C_0,\theta C_1) &= 0\end{aligned} \tag{2.2}$$

So, C_0 is a deformation of the Poisson superalgebra, and (2.2) is the equation, which we will investigate below for some of the deformations C_0, found in [11] and [12].

2.1 Antibracket

An interesting example of Lie superalgebra with 1 even and 1 odd cohomology is antibracket realized on D_n^n.

The spaces D_n^n and E_n^n possess also another $\mathbb{Z}_2$-grading ϵ (ϵ-parity), which is inverse to ε-parity: $\epsilon = \varepsilon + 1$.

We set $\varepsilon_A = 0$, $\epsilon_A = 1$ for $A = 1, \ldots, n_+$ and $\varepsilon_A = 1$, $\epsilon_A = 0$ for $A = n_+ + 1, \ldots, n_+ + n_-$.

It is well known, that the bracket

$$[f, g](z) = \sum_{i=1}^{n} \left(f(z) \frac{\overleftarrow{\partial}}{\partial x_i} \frac{\partial}{\partial \xi_i} g(z) - f(z) \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\partial}{\partial x_i} g(z) \right),$$

which we following to [14] call "antibracket" or "odd bracket", defines the structure of Lie superalgebra on the superspaces $D_n \stackrel{def}{=} D_n^n$ and $E_n \stackrel{def}{=} E_n^n$ with the ϵ-parity.

Indeed, $[f, g] = -(-1)^{\epsilon(f)\epsilon(g)}[g, f]$, $\epsilon([f, g]) = \epsilon(f) + \epsilon(g)$, and Jacobi identity is satisfied:

$$(-1)^{\epsilon(f)\epsilon(h)}[f, [g, h]] + (-1)^{\epsilon(g)\epsilon(f)}[g, [h, f]] + (-1)^{\epsilon(h)\epsilon(g)}[h, [f, g]] = 0, \quad f, g, h \in E_n.$$

Here these Lie superalgebras are called antiPoisson superalgebras.

The odd Poisson bracket play an important role in Lagrangian formulation of the quantum theory of the gauge fields, which is known as BV-formalism [14], [15] (see also [16]-[18], [5]). These odd bracket were introduced in physical literature in [14]. Antibracket possesses many features analogous to ones of even Poisson bracket and even can be obtained via "canonical formalism" with odd time. However, contrary to the case of even Poisson bracket where there exists voluminous literature on different aspects of the deformation (quantization) of Poisson algebra, the problem of the deformation of antibracket is not study satisfactory yet.

Theorem 2.1. [11] *Up to similarity transformation, the deformation of antiPoisson superalgebra with even parameter $\hbar$ has the form*

$$[f(z),\, g(z)]_* = [f(z), g(z)] + (-1)^{\varepsilon(f)}\{\frac{c}{1 + cN_z/2}\Delta f(z)\}\mathcal{E}_z g(z) + \{\mathcal{E}_z f(z)\}\frac{c}{1 + cN_z/2}\Delta g(z), \tag{2.3}$$

where $N_z = \sum_A z_A \frac{\partial}{\partial z_A}$, $\Delta = \sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial \xi_i}$ *and* $c \in \hbar^2\mathbb{K}[[\hbar^2]]$.

2.1.1 Odd cohomology of antiPoisson superalgebra and corresponding deformation

Odd cohomology of antiPoisson superalgebra has the form [19]

$$m_{2|3}(z|f, g) = (-1)^{\varepsilon(f)}\{(1 - N_\xi)f(z)\}(1 - N_\xi)g(z) \tag{2.4}$$

One can prove that in this case the equation (2.2) has the only solution which leads to the following deformation

$$[f, g]_\star = [f, g] + \theta\left(1 - \xi^i\partial_{\xi^i}\right) f \cdot \left(1 - \xi^i\partial_{\xi^i}\right) g$$

2.2 Deformations of the Poisson superalgebra.

For any $\varkappa \in \mathbb{K}[[\hbar]]$, such that $c_1 \stackrel{def}{=} \frac{1}{6}\hbar^2\varkappa^2 \in \hbar^2\mathbb{K}[[\hbar^2]]$, the Moyal-type superbracket

$$\mathcal{M}_{c_1}(z|f,g) = \frac{1}{\hbar\varkappa} f(z) \sinh\left(\hbar\varkappa \sum_{A,B} \frac{\overleftarrow{\partial}}{\partial z_A}\omega^{AB}\frac{\partial}{\partial z_B}\right) g(z)$$

is skew-symmetric and satisfies the Jacobi identity and, therefore, gives a deformation of the initial Poisson algebra.

Let the bilinear mappings m_3, and m_ζ from $(D^{n_-}_{n_+})^2$ to $D^{n_-}_{n_+}$ be defined by the relations

$$m_3(z|f,g) = (-1)^{n_- \varepsilon(f)}\mathcal{E}_z f(z)\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)+n_-\varepsilon(g)}\mathcal{E}_z g(z)\bar{f}, \tag{2.5}$$

$$m_\zeta(z|f,g) = (-1)^{n_- \varepsilon(f)}\{\zeta(z), f(z)\}\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)+n_-\varepsilon(g)}\{\zeta(z), g(z)\}\bar{f}, \tag{2.6}$$

where $\mathcal{E}_z \stackrel{def}{=} 1 - \frac{1}{2}z\partial_z$, and $z = (x_1, ..., x_n, \xi_1, ..., \xi_m)$.

For $\zeta \in E^{n_-}_{n_+}[[\hbar^2]]$, $c_1, c \in \mathbb{K}[[\hbar^2]]$, we set

$$C^{(1)}_{\zeta,c_1}(z|f,g) = \mathcal{M}_{c_1}(z|f+\zeta\bar{f}, g+\zeta\bar{g}),$$
$$C^{(1)}_{\zeta,c_1,c}(z|f,g) = \mathcal{M}_{c_1}(z|f+\zeta\bar{f}, g+\zeta\bar{g}) + c\bar{f}\bar{g}$$

2.2.1 Deformations of the Poisson superalgebra at $n_+ \geq 4$.

Theorem 2.2. [12] *Let $\theta_i = 0$ for all i. Then*

1. *Let $n_- = 2k$ and $n_+ \geq 4$. Then every continuous formal deformation of the Poisson superalgebra $D^{n_-}_{n_+}$ is equivalent either to the superbracket $C^{(1)}_{\zeta,c_1}(z|f,g)$, where $\zeta \in \hbar^2 E^{n_-}_{n_+}[[\hbar^2]]$ is even and $c_1 \in \hbar^2\mathbb{K}[[\hbar^2]]$, or to the superbracket*

   $$C^{(3)}_{\zeta,c_3}(z|f,g) = \{f(z), g(z)\} + m_\zeta(z|f,g) + c_3 m_3(z|f,g),$$

   where $\zeta \in \hbar^2 E^{n_-}_{n_+}[[\hbar^2]]$ is even and $c_3 \in \hbar^2\mathbb{K}[[\hbar^2]]$. The deformations $C^{(i)}_{\zeta_1,c}$ and $C^{(i)}_{\zeta_2,c}$ are equivalent if $\zeta_1 - \zeta_2 \in Z^{n_-}_{n_+}[[\hbar^2]]$.

2. *Let $n_- = 2k+1$ and $n_+ \geq 4$. Then every continuous formal deformation of the Poisson superalgebra $D^{n_-}_{n_+}$ is equivalent to the superbracket $C^{(1)}_{\zeta,c_1,c}(z|f,g)$, where $c, c_1 \in \hbar^2\mathbb{K}[[\hbar^2]]$ and $\zeta \in \hbar^2 E^{n_-}_{n_+}[[\hbar^2]]$ is an odd function such that $[\mathcal{M}_{c_1}(z|\zeta,\zeta) + c] \in D^{n_-}_{n_+}[[\hbar^2]]$. The deformations $C^{(1)}_{\zeta_1,c_1,c}$ and $C^{(1)}_{\zeta_2,c_1,c}$ are equivalent if $\zeta_1 - \zeta_2 \in Z^{n_-}_{n_+}[[\hbar^2]]$.*

2.2.2 Odd cohomology

1. Let $n_- = 2k$ and $n_+ \geq 4$.

   Then the odd cohomology has the form

   $$m_\zeta(z|f,g) \tag{2.7}$$

   with odd function ζ.

2. Let $n_- = 2k+1$ and $n_+ \geq 4$.

   Then the odd cohomology has either the form

$$m_\zeta(z|f,g) \tag{2.8}$$

   with even function ζ or the form

$$m_3(z|f,g). \tag{2.9}$$

2.2.3 Deformation of Poisson superalgebra for even n_-

Theorem 2.3. *Let n_- be even and $n_+ \geq 4$. Then each deformation of Poisson superalgebra with one odd deformation parameter is equivalent either to*

$$\{f,g\}_\star = \frac{1}{\hbar\kappa}(f+\zeta\bar{f})\sinh\left(\hbar\kappa\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}\right)(g+\zeta\bar{g})$$

or to

$$\{f,g\}_\star = \{f,g\} + m_\zeta(f,g) + cm_3(f,g),$$

where $\zeta = \zeta_0 + \theta\zeta_1,\ \zeta_i \in \hbar^2 S^{n_-}_{n_+}[[\hbar^2]],\ \epsilon(\zeta_i) = i$ *and* $\kappa^2,\ c \in \hbar^2\mathbf{C}[[\hbar^2]]$.

This Theorem can be proved analogously to Theorem 2.2.

2.2.4 Deformation of Poisson superalgebra for odd n_-

Theorem 2.4. *Let n_- be odd and $n_+ \geq 4$. Then each deformation of Poisson algebra with one odd deformation parameter is equivalent either to*

$$\{f,g\}_\star = \frac{1}{\hbar\kappa}(f+\zeta\bar{f})\sinh\left(\hbar\kappa\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}\right)(g+\zeta\bar{g}) + c\bar{f}\bar{g}$$

or to

$$\{f,g\}_\star = \{f,g\} + m_\zeta(f,g) + \theta m_3(f,g),$$

where

$$\zeta = \zeta_1 + \theta\zeta_0,\ \zeta_i \in \hbar^2 S^{n_-}_{n_+}[[\hbar^2]],\ \epsilon(\zeta_i) = i,$$
$$\kappa^2, c \in \hbar^2\mathbf{C}[[\hbar^2]],$$

are such that

$$\frac{1}{\hbar\kappa}\zeta\sinh\left(\hbar\kappa\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}\right)\zeta + c \in \mathbf{D}[[\hbar^2]].$$

This Theorem can be proved analogously to Theorem 2.2.

3 Finite number of odd deformation parameters

Let us look for deformation of Poisson superalgebra with more than 1 odd deformation parameters.

3.1 Deformation of Poisson superalgebra for even n_-

In the case of even n_- we can reduce some number of odd cohomologies $m_{\zeta_{\alpha_1,\dots,\alpha_k}}$ ($\alpha_i = 0, 1$) with odd $\zeta_{\alpha_1,\dots,\alpha_k} \in S^{n_-}_{n_+}$ to even m_ζ with even $\zeta \in \mathbf{S}$:

$$\zeta = \sum_s \sum_{\alpha_1,\dots,\alpha_k=0,1} (\theta_1)^{\alpha_1} \cdot \ldots \cdot (\theta_k)^{\alpha_k} \cdot \zeta_{\alpha_1,\dots,\alpha_k}, \quad \varepsilon(\zeta) = 0 \tag{3.1}$$

Then one can easily prove the following Theorem, which is complete analog of the first item of Theorem 2.2.

Theorem 3.1. *Let $n_- = 2k$ and $n_+ \geq 4$. Then every continuous formal deformation of the Poisson superalgebra $\mathbf{D}$ is equivalent either to the superbracket $C^{(1)}_{\zeta,c_1}(z|f,g)$, where $\zeta \in \hbar^2\mathbf{D}[[\hbar^2]]$ is even and $c_1 \in \hbar^2\mathbb{G}^k[[\hbar^2]]$ is even, or to the superbracket*

$$C^{(3)}_{\zeta,c_3}(z|f,g) = \{f(z), g(z)\} + m_\zeta(z|f,g) + c_3 m_3(z|f,g),$$

where $\zeta \in \hbar^2\mathbf{E}[[\hbar^2]]$ is even and $c_3 \in \hbar^2\mathbb{G}^k[[\hbar^2]]$ is even. The deformations $C^{(i)}_{\zeta_1,c}$ and $C^{(i)}_{\zeta_2,c}$ are equivalent if $\zeta_1 - \zeta_2 \in \mathbf{Z}[[\hbar^2]]$.

3.2 Deformation of Poisson superalgebra for odd n_-

There are infinite number of odd cohomologies in this case also: m_3 and m_ζ with even $\zeta \in \S^{n_+}_{n_-}$. We can reduce some finite part of these odd forms to even by multiplying them by odd elements of $\mathbb{G}^k$: $m_3 \to \theta_1 m_3$, $m_{\zeta_{\alpha_1,\dots,\alpha_k} \in S^{n_-}_{n_+}} \to m_\zeta$ with odd $\zeta \in \mathbf{S}$:

$$\zeta = \sum_s \sum_{\alpha_1,\dots,\alpha_k=0,1} (\theta_1)^{\alpha_1} \cdot \ldots \cdot (\theta_k)^{\alpha_k} \cdot \zeta_{\alpha_1,\dots,\alpha_k}, \quad \varepsilon(\zeta) = 1 \tag{3.2}$$

Let us look for the deformation. Consider the zero and first order terms:

$$C(f,g) = m_0(f,g) + hm_1(f,g) + \theta_1 m_3(f,g) + m_\zeta(f,g) + (\textit{higher order terms}).$$

Jacobi identity gives the relation $\theta_1 h = 0$.

We restrict ourselves here to the case $\theta_1 \neq 0$, which implies $h = \theta_1 h_1$. Below we omit everywhere the subscript 1 at θ_1.

The following decomposition orders lead to the next form of the deformation

$$C = m_0 + \theta h_1 m_1 + \theta m_3 + m_\zeta + \theta h_1 j_\zeta + \eta(z)\mu,$$

where

$$j_\zeta(f,g) = (-1)^{n_-\varepsilon(f)} m_1(\zeta,f)\overline{g} - (-1)^{\varepsilon(f)\varepsilon(g)+n_-\varepsilon(g)} m_1(\zeta,g)\overline{f}$$
$$\mu(f,g) = (-1)^{\varepsilon(f)}\overline{f}\overline{g},\ \varepsilon(\mu) = 0,$$

and even function $\eta(z) \in \mathbf{D}$ should be determine.

The Jacobi identity gives the following relations

$$\begin{aligned}
&\{\zeta,\eta\}+\bar\eta\eta+\theta[\mathcal{E}-(2+n_+-n_-)]\eta+hm_1(\zeta,\eta)=0,\\
&\theta h=0,\quad \theta\bar\eta=0,\\
&hm_1(\{\zeta,\zeta\},f(z))+hm_1(\eta,f(z))+h\bar\eta m_1(\zeta,f(z))=0,\\
&\eta+hm_1(\zeta,\zeta)+\theta[2\mathcal{E}-(2+n_+-n_-)]\zeta+\bar\eta\zeta+\{\zeta,\zeta\}=h_2,
\end{aligned}$$

where $h_2\in\mathbb{G}^k$ is even.

This system of relations is equivalent to the following relations

$$\begin{aligned}
&\eta=-\theta h_1m_1(\zeta,\zeta)-\theta[2\mathcal{E}-(2+n_+-n_-)]\zeta-\bar\eta\zeta-\{\zeta,\zeta\}+h_2,\\
&\theta\bar\eta=0,\\
&\theta(1+n_+-n_-)h_2-\bar\eta h_2=0.
\end{aligned}$$

Thus, we've obtained theorem

Theorem 3.2. *Let n_- be odd and $n_+\geq 4$. Then Poisson superalgebra has the deformation depending on k odd parameters*

$$C=m_0+\theta h_1m_1+\theta m_3+m_\zeta+\theta h_1 j_\zeta+\eta(z)\mu,$$

where $\zeta\in\mathbf{S}$, $\eta\in\mathbf{D}$ and h_1, $h_2\in\mathbb{G}^k$ satisfy the relations

$$\begin{aligned}
&\eta=-\theta h_1m_1(\zeta,\zeta)-\theta[2\mathcal{E}-(2+n_+-n_-)]\zeta-\bar\eta\zeta-\{\zeta,\zeta\}+h_2,\\
&\theta\bar\eta=0,\\
&\theta(1+n_+-n_-)h_2-\bar\eta h_2=0,\\
&\varepsilon(z)=1,\ \varepsilon(\eta)=0,\ \varepsilon(h_1)=1,\ \varepsilon(h_2)=0.
\end{aligned}$$

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